touching are those whose extreme limits have the same position, things successive are those between which there lies nothing that is like them, then it is impossible that a thing continuous consist of indivisible things, as for instance, that a line be made up of points, the line being taken as continuous, and a point as indivisible."

This passage is made clearer by a quotation or two from the *Physics*, Bk. V, § 3:

"Continuous are those things which in their union become one in nature; and just as that which holds them in continuity together becomes one, so the whole will be one, as for instance by means of a nail, or glue, or an adherence or an accretion."

"Things continuous must necessarily touch; on the other hand, things touching are not necessarily continuous, for the extreme ends, even though locally together, are not necessarily one and the same."

Returning to the *Physics*, Bk. VI, § 1, we reproduce the chief part of Aristotle's argument which leads him to the conclusion that a line is not made up of points:

"For neither are the extreme ends of points one [united], for the reason that, of an indivisible, the one cannot be an extreme and the other another part, nor are the extreme ends locally together, since the indivisible can have no extreme."

According to Aristotle, things continuous are always divisible into parts that are continuous. In the same way, time is not made up of parts considered as indivisible *Nows*.[1] We quote from *Physics*, Bk. VI, § 2:

"As each magnitude is divisible into magnitudes (for it has just been proved that the continuous cannot consist of the indivisible, and every magnitude is continuous), the faster of two must necessarily in the same time be moved through a larger distance and in a less time through an equal distance, and also in a less time through a larger distance, as indeed some define the faster. . . ."

"Since all motion takes place in time, and in each time it is possible for something to be moved and everything movable can be moved faster and also slower, then there can occur in each time the faster and the slower motion. If this is so, then time must be continuous; by the continuous I mean that which is always divisible into divisible parts. . . ."

"If time is continuous, so is distance, for in half the time a thing passes over half the distance, and, in general, in the smaller time the smaller distance, for time and distance have the same divisions; and if one of the two is unlimited, so is the other. . . . For that reason the argument of Zeno assumes an untruth, that one unlimited cannot travel over another unlimited along its own parts, or touch such an unlimited, in a finite time; for length as well as time and, in general, everything continuous, may be considered unlimited in a double sense, namely, according to the [number of] divisions or according to the [distances between the] outermost ends."

This profound criticism directed against Zeno refers evidently to the "Achilles" where we are in danger of modifying, in our minds, the conditions actually existing; we are in danger of thinking of the distance between Achilles and the tortoise as decreasing through an unlimited number of subdivisions, while the time for traversing these successive subdivisions is thought of as about the same for each. There results from this distortion of events an unlimited subdivision of a finite distance, and an unlimited accumulation of finite time-intervals; the former yields what from a modern view-point constitutes a convergent geometric series of space-intervals; the latter a divergent series of time-intervals. A finite distance is made to be traversed in an infinite time.

Aristotle's arguments against the "Achilles" and the "Dichotomy" are the

[1] See *Physics*, Bk. VI, § 9, quoted earlier in this article.

same. He touches upon them in other parts of his *Physics*. Since a line cannot be built up from points, a line cannot actually be subdivided into points.[1]

"The continued bisection of a quantity is unlimited, so that the unlimited exists potentially, but is actually never reached."[2]

"Previously we refuted this [the "Dichotomy"] by the fact that time has unlimitedly many parts, in consequence of which there is no absurdity in the consideration that in unlimited time-intervals one passes over unlimitedly many spaces . . . [but as Aristotle does not consider this a full explanation, he continues:] If one divides a continuous line into halves, he uses one point at two, for he makes it the beginning [of one part] and the end [of the other]; in this manner proceeds he who . . . bisects, but in this division neither the line nor the motion are continuous; for . . . in the continuous there are, to be sure, unlimitedly many halves, but not actually unlimited, only potentially so."[3]

Of the numerous passages in Aristotle's *Physics* which might be quoted as bearing on the "Arrow" we choose the following (Bk. VI, § 8):

". . . A thing is at rest, when it is unchanged in one *Now* and still in another *Now*, it itself as well as its parts remaining in the same status. . . . There is no motion nor rest in the *Now* . . . In a time-interval, on the contrary, it [a variable] cannot exist in the same state of rest, for otherwise it would follow that the thing in motion is at rest."

Aristotle's disproof of the "Stade" is given at the end of our first extract from the *Physics* (Bk. VI, § 9), where it is pointed out that Zeno falsely assumes a body to move with the same velocity relative to a body that is at rest, as it would with respect to one in motion.

Further Comments on Aristotle. The main criticism to be made on the profound arguments of Aristotle is that they do not go far enough. The "Dichotomy" and "Achilles" involve the theory of limits, a theory which in very recent time has been found to be in need of reconstruction and which was imperfectly developed in Greek antiquity. Particularly insistent in our mind is the query raised by Zeno's arguments, *how is it possible for a variable to reach its limit?* This query finds no reply in Aristotle. To be noted is also the fact that Aristotle denied the existence of actual infinity, as distinguished from potential infinity. The "Arrow" called for a sharp definition of the *Now* (the instant). Has the instant no duration? Is it, so to speak, a point of time? Or is the instant an infinitesimal—some constant different from zero, yet smaller than any finite quantity? Here Aristotle drew a sharp line; the *Now* was a point of time; it had no duration; in the *Now* there could be neither motion nor rest.

It will be seen that the mathematical concepts herein involved are most fundamental. The concept of a limit involves notions of infinity and is historically connected with the concept of the infinitesimal. All these concepts are basic. It would be difficult to select three other notions more far-reaching in mathematical science than *infinitesimal, infinity, limit.*

[1] Aristotle, *Physik*, VI, 1, Prantl's ed., p. 281.

[2] Aristotle, *Physik*, III, 7, Prantl's ed., p. 141.

[3] Aristotle, *Physik*, VIII, 8, Prantl's ed., pp. 445, 447. See also Aristotle's *Lib de lineis insecab.*, p. 968.

III.

C. A Two-Thousand Year Struggle for Light.

1. The Greeks after Aristotle.

Whether Plato and Aristotle correctly explained the nature of Zeno's arguments and the purpose which Zeno himself had in mind in presenting them, is of no concern in tracing the history of thought on this subject after the time of Aristotle. Apparently the writings of Plato and Aristotle constituted the sources of information for later writers. Zeno's arguments were given by Aristotle in the form of "fallacies," and it is the influence of these "fallacies" that remains to be traced. There is little doubt that this influence was great upon the development of Greek geometry. Since Aristotle, with all his dialectical skill, was not able to satisfactorily explain all the paradoxes which had arisen in the study of the infinite and of motion, the conclusion of H. Hankel[1] and other recent historians of mathematics is probably correct, that the infinite and infinitesimal were banished from the classic Greek geometry for the sake of greater rigor. We shall see that recently discovered manuscripts of Archimedes confirm this view. Mathematicians assumed that every magnitude is divisible at pleasure. The doctrine of incommensurable lines rests upon the possibility of unlimited divisibility. The denial of the existence of the infinitesimal goes back to Zeno who is reported by Simplicius[2] to have stated: "That which, being added to another, does not make it greater, and being taken away from another does not make it less, is nothing." This momentous question presented itself twenty-two centuries later to Leibniz who gave different answers. In one exposition Leibniz extended the definition of equality so as to declare magnitudes as equal when they differ from one another by an incomparably small quantity. The later Greek mathematicians followed a radical policy toward the infinitesimal; they formally excluded it from demonstrative geometry by a postulate. This was done by Eudoxus (408–355 B. C.), by Euclid (about 300 B. C.) and by Archimedes (287–212 B. C.). Archimedes gives the postulate, which he attributes to Eudoxus, as follows:[3]

> "When two spaces are unequal, it is possible to add to itself the difference by which the lesser is surpassed by the greater, so often that every finite space will be exceeded."

Euclid in his *Elements* (Bk. V, Def. 4) gives the postulate in the form of a definition:

> "Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other."

The Method of Archimedes, a book formerly thought to be irretrievably lost, but fortunately discovered by Heiberg in 1906 in Constantinople, gives inter-

[1] H. Hankel, *Geschichte der Mathematik im Alterthum und Mittelalter*, Leipzig, 1874, p. 120.
[2] Simplicius, *Phys.* 30*a*. Quoted by E. Zeller, *History of Greek Philosophy*, Vol. 1, London, 1881, p. 615.
[3] Archimedes, *De quadr. parabol. Praef.*

esting evidence that the notion of infinitesimals, though not used by Archimedes in formal demonstrations, was employed by him in tentative research as a method of discovery. He considered infinitesimals sufficiently scientific to suggest the truths of theorems, but not to furnish rigorous proofs. The process is mechanical, consisting of the weighing of infinitesimal elements, which he calls straight lines or plane areas, but which are really infinitely narrow strips or infinitely thin plane laminæ.[1] The breadth or thickness is regarded as being the same in the elements weighed at any one time.

Further evidence that infinitesimals lingered in the minds of Greek thinkers through the centuries is furnished by the skeptic, Sextus Empiricus (200 A. D.), who advances the paradox that, when a line rotating in a plane about one of its ends describes a circle with each of its points, these concentric circles are of unequal area, yet each circle must be equal to the neighboring circle which it touches.[2] The difficulty encountered here is similar to that raised by Democritus over 500 years earlier.

An interesting remark about Zeno is made by Plutarch (about 90 A. D.) in his life of Pericles. He says that Pericles was also a hearer of Zeno, the Eleatic, who "also perfected himself in an art of his own for refuting and silencing opponents in argument; as Timon of Phlius describes it—

> Also the two-edged tongue of mighty Zeno, who,
> Say what one would, could argue it untrue."

Thus we see that the personality of Zeno and some of the ideas involved in his "paradoxes" appear here and there on the surface of Greek thought, thereby indicating an underground flow of those ideas down the centuries of Greek history.

Sextus Empiricus also gives a version of the "Arrow" much like what we quoted from Diogenes Laertius. He attributes the paradox not to Zeno but to Diodorus Cronus. The existence of motion is disproved thus: If matter moves, it is either in the place in which it is, or the place in which it is not; but it cannot move in the place in which it is, and certainly not in the place in which it is not: hence it cannot move at all. To this Sextus Empiricus replies by stating another argument equally paradoxical and therefore far from illuminating: By the same rule men never die, for if a man die, it must either be at a time when he is alive, or at a time when he is not alive; hence he never dies.

2. The Romans.

The subtleties of Zeno's arguments on motion attracted little attention among the Romans. Lucretius (96–55 B. C.) used the notion of infinity in arguments on atomic theory. He reasoned that one must assume the existence of atoms (indivisible, but not mathematical points), otherwise each body,

[1] T. L. Heath, *Method of Archimedes*, Cambridge, 1912, p. 8.

[2] Sextus Empiricus, *Adv. math.*, I, III, § 66 ff., ed. Fabricius. p. 322; referred to in K. Lasswitz, *Geschichte der Atomistik* I, Hamburg u. Leipzig, 1890, p. 148.

whether large or small, would consist of an infinite number of parts, and there would be no difference between the largest and the smallest, both being infinity.[1]

Cicero and Seneca mention Zeno in passing: the one to display Zeno's love for argument even though faulty, the other to display Zeno's skepticism. Cicero[2] attributes to Zeno the following pointed syllogism: "That which exercises reason is more excellent than that which does not exercise reason; there is nothing more excellent than the universe; therefore the universe exercises reason." Seneca[3] exhibits Zeno as denying not only plurality as did his master, but denying also unity and the real existence of external objects. We have no good reason for accepting either Cicero's or Seneca's view of Zeno's aim and purpose.

3. Medieval Times.

The earliest church father known to interest himself in arguments on motion was St. Augustine (354–430 A. D.). In a dialogue on the question, whether or not the mind of man moves when the body moves, and travels with the body, he is led to a definition of motion, in which he displays some levity. It has been said of scholasticism that it had no sense of humor. Hardly does this apply to St. Augustine in his discussion of the impossibility of motion. He says:

> "When this discourse was concluded, a boy came running from the house to call us to dinner. I then remarked that this boy compels us not only to define motion, but to see it before our very eyes. So let us go, and pass from this place to another; for that is, if I am not mistaken, nothing else than motion."[4]

St. Augustine deserves also the credit of having accepted the existence of the *actually infinite* and to have recognized it as being, not a variable, but a constant. He recognized *all* finite positive integers as an infinity of that type.[5] On this point St. Augustine occupied a radically different and more advanced position than his forerunner, Origen of Alexandria; who took a decided stand against the actually infinite and supported his position by arguments which G. Cantor admits to be the profoundest ever advanced against the actually infinite.[6] In their completest form, these arguments were given many centuries later by the great Italian philosopher of the Middle Ages, Thomas Aquinas (1225(?)–1274).[7] Of importance to us is the nature of the continuum, particularly the linear continuum, as described by Aquinas. It was conceived as *potentially* divisible to infinity, since practically the divisions could not be carried out to infinity. There was, therefore, no minimum line. On the other hand, the point is not a constituent part of a line, since it does not possess the property of infinite divisi-

[1] Lucretius, *De rerum natura*, ed. I. Bernays, Leipzig, 1886, I, 615 ff.
[2] *De natura deorum*, Book III, IX.
[3] Epistola 88.
[4] St. Augustine, *De ordine*, II, VI, 18.
[5] S. Augustin, *De civitate Dei*, lib. XII, cap. 19. The chapter is quoted by G. Cantor in "Mitteilungen zur Lehre vom Transfiniten," *Zeitschr. f. Philosophie u. philosoph. Kritik*, Bd. 91, p. 81. Separatabdruck, p. 32.
[6] G. Cantor, *op. cit.*, Separatabdruck, p. 35.
[7] Thomas Aquinas, *Summa theol.*, I, q. 7 a. 4. Quoted by G. Cantor, *op. cit.*, Separatabdruck, p. 36.

bility that parts of a line possess, nor can the continuum be constructed out of points. However, a point by its motion has the capacity of generating a line.[1]

This concept of the continuum, as held by Aquinas, is a fair representation of the prevalent medieval scholastic views on this topic. It held a firm ascendancy over the ancient atomistic doctrine which assumed matter to be composed of very small, indivisible particles, possessing the properties of matter itself. No continuum superior to it was created before the nineteenth century.

In his commentaries on Aristotle's *Physics*, Aquinas[2] explains at some length the arguments of Zeno against motion as they are given by Aristotle. Aquinas shows a complete mastery of the subject as expounded by the Stagirite, but hardly presents any new points of view.[3]

Early English Writers. The earliest Englishman known to have written on continuity and infinity is Roger Bacon (1214(?)–1294), the seven-hundredth anniversary of whose birth was celebrated at Oxford in 1914. Bacon argued against the composition of the continuum of *indivisible* parts (different from points), by renewing the arguments presented by the Greeks and the early Arabs. Bacon held that the hypothesis of indivisible parts of uniform size would make the diagonal of a square commensurable with a side; if the ends of an indivisible part of a circle are connected by radii with the center of the circle, then the two radii would intercept an arc on a concentric circle of smaller radius. From this it would follow that the inner circle is of the same length as the outer circle. This is impossible. Bacon argued also against infinity. If time were infinite, it would follow that the part is equal to the whole—a deduction which he considered absurd. Similar arguments lead him to conclude that the world is finite.[4]

The views of Roger Bacon became known more widely through Duns Scotus (1265–1308), the theological and philosophical opponent of Thomas Aquinas. However, Scotus and Aquinas took the same ground in teaching that in the continuum there existed actual, indivisible points. Thereby it is not admitted that the continuum is made up of, or consists wholly of, points; the indivisible

[1] C. R. Wallner, in *Bibliotheca mathematica*, 3. F., Bd. IV, 1903, pp. 29, 30, gives quotations from Thomas Aquinas, *Opuscula omnia*, 1562, o. 52, p. 369; o. 36, c. 2; o. 44, c. 1; o. 44, c. 2, p. 280.

[2] *Opera omnia*, Tom. II, Pars prima: *Sancti Thomae aqvinatis ex ordine praedicatorum quinti ecclesiae doctoris angelici praeclarissima commentaria in octo Physicorum Aristotelis libros. . . . Ad haec accessit Roberti Linconiensis in eosdem summa.* Parisiis, MDCLX, Lectio XI, pp. 233–237, 352.

[3] A summary of what is given in each of Aristotle's eight books on Physics is given by Robert Grosseteste (1175(?)–1253), bishop of Lincoln, whom Roger Bacon praises as a scientist of high rank. The esteem in which his writings were held appears from the fact that his summary of Aristotle is reproduced four centuries after his death in an edition of Aquinas. Of Zeno's arguments Grosseteste says (page 352): *Ad primum dicitur (sicut prius dictum est id dubitatione præcedenti) scilicet quòd continuum est infinitum secundum potentiam & tale potest transiri. Et sic patet ad secundam rationem. Ad tertium dicitur quòd Zeno dixit ipsum tempus componi ex instantibus, quod non est verum, ideò nec motus nec quies est in instanti, sed in tempore [sicut dixit Philosophus] ideò mobile non est spacio sibi aequali nisi tantum in instanti. Et cum dicitur, aut movetur, aut quiescit, negatur propositio, quia habet veritatem de es quod est in aliquo in quo aptum natum moveri aut quiescere pro tali mensura temporis.*

[4] See *Opera hactenus inedita Baconi*, Fasc. 1, Metaphysica. Edidit Robert Steele. London, p. 11; Jonas Cohn, *Geschichte des Unendlichkeitsproblems*, Leipzig, 1896, pp. 76, 77; K. Lasswitz, *op. cit.*, Vol. I, pp. 193, 195.

points might, for instance, be simply end points. These contentions are directed against the atomists. The arguments are wanting in explicitness and precision. What we said of Aquinas's commentary on Zeno applies also to Duns Scotus.[1] He gives detailed elaborations of Aristotle without offering new explanations of Zeno's puzzles. In place of Achilles and the tortoise he introduces the more familiar travelers, the horse and the ant. His commentaries are annotated by the Franciscan theologian Franciscus de Pitigianis of Arezzo in Italy, who wrote the latter part of the sixteenth century. This annotator expresses himself in favor of the admission of the actual infinity to explain the "Dichotomy" and the "Achilles," but fails to adequately elaborate the subject. Scholastic ideas on infinity and the continuum find expression in the writings of Bradwardine, the English *doctor profundus*. He says that five explanations have been given of the nature of the continuum.[2]

GROUPS OF SUBTRACTION AND DIVISION WITH RESPECT TO A MODULUS.

By G. A. MILLER, University of Illinois.

Certain kinds of groups of subtraction and division were explained by the present writer in two articles entitled: "Groups of the fundamental operations of arithmetic" and "Groups of subtraction and division." These articles were published respectively in the *Annals of Mathematics*, volume 6 (1905), page 41; and in the *Quarterly Journal of Mathematics*, volume 37 (1906), page 80. The present article is devoted to more elementary considerations, and has for its main object to exhibit interesting elementary relations between certain groups of subtraction and division, and the corresponding groups of addition and multiplication.

It is well known, and also evident, that the first $m-1$ natural numbers together with zero constitute the cyclic group of order m with respect to addition when the sums are replaced by their least positive residues, or by zero, modulo m. That is, if in the series of numbers

$$0,\ 1,\ 2,\ \cdots,\ m-1$$

each number is replaced by itself increased by α, mod m, where $0 \equiv \alpha \equiv m-1$, there results a certain substitution on these m numbers, and the totality of the distinct substitutions which can be constructed in this manner constitutes the cyclic group of order m. The order of the substitution corresponding to α is

[1] Duns Scoti, *Opera Omnia*, T. II: Joannis Duns Scoti Doctoris Subtilis, ordinis minorum, in VIII libros Physicorum Aristotelis Quæstiones, cum annotationibus R. P. F. Francisci Pitigiani arretini, etc. Lvgdvni, MDCXXXIX, Quaestio X, pp. 390–393.

[2] See Maximilian Curtze on the "Tractatus de continuo Bradwardini" in *Zeitschrift f. math. u. Phys.*, XIII Jahrg., Suppl., 1868, Leipzig, p. 88.

evidently the quotient obtained by dividing m by the highest common factor of α and m. In particular, when $\alpha = 0$ this substitution reduces to the identity.

If each of the numbers of the given series is replaced by itself decreased by α, mod m, there results again a substitution, and the totality of the distinct substitutions which can be obtained in this manner constitutes again the cyclic group of order m. In fact, the substitution which corresponds to α when α is subtracted from each of the given numbers is the inverse of the substitution which corresponds to α when it is added to each of these numbers, since the successive performance of these two operations leaves each of the given numbers unchanged. As an automorphism of any abelian group can be established by letting each operator of this group correspond to its inverse, it results that a simple isomorphism between the given group of addition and the given group of subtraction can be established in such a way that the operations which result from the same number in the processes of addition and subtraction correspond in this simple isomorphism.

The simple isomorphism which has been considered exhibits very clearly that the operations of subtracting successively each of the numbers of the given series from every number of this series have the same relative properties as those of adding these same numbers to every number of this series. The fact that each operator corresponds to its inverse in the given simple isomorphism may be regarded as an extension of the concept that addition and subtraction are inverse operations. In fact, not only are these operators inverses but the two groups which they constitute under the given conditions are such that every operator of the one corresponds to the inverse of the other.

When the $\phi(m)$ natural numbers which do not exceed m and are prime to m are combined by multiplication, mod m, they form one of the most important classes of abelian groups. For any particular value of m, the corresponding group of order $\varphi(m)$ can be obtained by replacing these $\phi(m)$ numbers by the set obtained by multiplying each of them by k, provided k is prime to m and mod m is replaced by mod km. In fact, it has been observed that a necessary and sufficient condition that a series of distinct natural numbers constitutes a group as regards multiplication, mod m, is that each of the numbers of the series has the same highest common factor with m and that the quotient obtained by dividing m by this highest common factor is prime to this factor.[1] The special but fundamental case when this highest common factor is unity is the one which is generally treated in the text-books.

In this special case it is very easy to see that the same group of order $\varphi(m)$ results if the operation of multiplication is replaced by that of division, when the quotient α/β, mod m, is defined as usual, as an integer γ such that $\beta\gamma \equiv \alpha$, mod m, α and β being natural numbers. The substitution on the given $\varphi(m)$ numbers which results if all of these numbers are multiplied by any one of them, for instance α, is clearly the inverse of the one which results when all of these numbers

[1] *Annals of Mathematics*, series 2, vol. 6 (1905), p. 44. It may be observed that the statement of this theorem in the *American Journal of Mathematics*, vol. 27 (1905), p. 315, is inaccurate.

are divided by α. Hence the two substitutions which result from operating with the same number correspond in one of the possible simple isomorphisms between the given groups of multiplication and division, mod m.

In the more general case noted above, when a series of distinct natural numbers constitute a group as regards multiplication, we cannot always pass directly to a group of division without further restrictions. In fact, this more general set of numbers does not include unity with respect to the modulus, while the division of a number by itself gives unity for a quotient. It is, however, easy to see that whenever a set of numbers constitutes a group as regards multiplication, mod n, these numbers must also constitute a group as regards division, mod n, provided the quotient is restricted to the given set of numbers mod n.

As the converse of this proposition is evidently also true it results that *a necessary and sufficient condition that a set of distinct natural numbers constitutes a group with respect to division, mod m, when all of these numbers are divided successively by each one of them and the quotients are restricted to numbers of the set, is that m has the same highest common factor with each of these numbers and that the quotient obtained by dividing m by this factor is prime to this factor.* For instance, the set of numbers; 2, 4, 8, 10, 14, 16 constitutes the cyclic group of order 6 with respect to each of the operations of multiplication and division mod 18, provided that in the latter case the quotients are restricted to this set of numbers. In general, the groups of multiplication and division which are obtained in this manner are simply isomorphic, and the substitutions which correspond to the same number in these two groups are the inverses of each other.

CALIFORNIA TEACHERS OF MATHEMATICS.

The following extracts from the report of the last annual meeting of the Mathematics Section of the California High School Teachers' Association by the chairman, Professor Henry W. Stager, of Fresno Junior College, are significant in many ways, and will be of interest to readers of the MONTHLY:

At the annual meeting in July, 1914, the Mathematics Section of the California High School Teachers' Association adopted an official reading course for the present school year. The purpose of this course is to arouse a greater interest in the subject on the part of teachers of mathematics rather than to increase their knowledge of mere mechanical methods of presentation. Every teacher is urged to undertake the careful reading of one or more of these books this year. The course is divided into three sections, graded according to difficulty. Each teacher can find some book suited to his individual needs. The University of California will grant credit for the study of certain of the books as part of the work in university extension. Detailed information may be obtained by addressing the University Extension Division, University of California, and referring to the course herewith.

Teachers are urged to ask their trustees to place the entire list in the school

library at the earliest possible opportunity. The list is also suitable for public libraries,—in fact it forms a most excellent list for the general reader who is interested in mathematics and desires an introduction to the modern view-point. Teachers are asked to suggest the purchase of these books by the local public libraries. The list which follows has been recommended only after the most careful consideration by the committee in charge.

SECTION I. Books of easy grade.

1. Ball: *A Primer of the History of Mathematics.*
2. Cajori: *A History of Mathematics.*
3. Smith-Karpinski: *The Hindu-Arabic Numerals.*
4. Whitehead: *An Introduction to Mathematics.*

SECTION II. Books of medium grade.

5. Ball: *Mathematical Recreations and Essays.*
6. Beman and Smith: *Klein's Famous Problems in Elementary Geometry.*
7. J. W. Young: *Fundamental Concepts of Algebra and Geometry.*
8. Fine: *The Number-System of Algebra.*

SECTION III. Books of more advanced grade.

9. J. W. A. Young, Editor: *Monographs on Modern Mathematics.*

In connection with the above, the committee suggest the reading of a good mathematical journal and recommend for this purpose, THE AMERICAN MATHEMATICAL MONTHLY.

In his report, the chairman stated that he had sent out two letters to each of the 400 teachers of mathematics in the state, calling their attention to the study plan adopted by the Section in 1913. The work of the Section had also been presented at the four sectional meetings of the California Teachers' Association. As far as could be judged from correspondence and interviews with teachers, there seemed to be a new and increased interest in the better teaching of mathematics throughout the state. A message from Professor H. E. Slaught, managing editor of THE AMERICAN MATHEMATICAL MONTHLY, was read:

"Press on in the way you have started. Show the country what can be done by a body of teachers who have ambition to move *onward and upward*, who believe that the best enrichment of the secondary field depends upon the strongest and highest development of the teachers themselves, not simply through activity on the *dead level of daily routine*, but also, and most emphatically, through personal power gained in the higher ranges of thought and study. We are just beginning to realize the importance of this matter in this country, though other countries, and especially Germany, have given us object lessons in plenty. But this is part of the general uplift in our whole educational system and it will spread rapidly when it is once understood. In fact, this is the only explanation of the wonderful growth of the summer school idea in the universities. Thousands of teachers are resolving to take part in the uplift by *uplifting themselves through higher study and training.* To be in the lead in this great movement is no small honor. Hence, God speed the California teachers in this "forward look."

The chairman advocated a continuous policy in the work of the Section, especially for the coming year. To this end the previous committee on ways and means was reappointed, consisting of Professor D. N. Lehmer, chairman, Miss Thirmuthis Brookman, of San Mateo, and Miss Sadie L. Gilmore, of Colusa, together with the chairman of the Section.

Two of the papers read concerned the administration of the intermediate schools, sometimes called junior high schools, consisting of the seventh, eighth, and ninth grades.

Miss Thirmuthis Brookman, for some years head of the mathematics department of the Berkeley high schools, insisted that all mathematics must be correlated with the life of the students. She illustrated her principles by applications of the arithmetic of investment and expenditure, civic arithmetic, and the like, as worked out in Berkeley for the first two years of the intermediate school. The last year was devoted to algebra, which was made extremely practical, by the alternation of the work on the abstract principles of algebra with chapters dealing with the applications of proportion, such as belted pulleys in sewing machines, gears in mesh, levers, etc. The natural focus of the ninth-grade work was the mastery of the quadratic formula.

Mr. Will C. Wood, California commissioner of secondary schools, showed that mathematics should have a place in this curriculum, from both a *practical* and a *disciplinary* standpoint. Mathematics should be required throughout the three years, leading up to a mastery of arithmetic and the beginning of algebra in the eighth year. The ninth year should complete the essentials of elementary algebra and include some of the fundamental ideas of geometry, such as parallel lines, similarity, and simple geometric constructions, all woven together into a unified whole. The final topic which he considered was the preparation and certification of the teachers for this school. A special preparation will be necessary for this work, involving more time and study than that required for elementary school teachers, but not so highly specialized as the teacher of the secondary school. New methods of teaching should be effected in which the *individual* is the aim of the teacher's thought rather than *uniformity*.

Professor D. N. Lehmer, of the University of California, spoke with great effect on "How shall the isolated teacher of mathematics keep up his interest." He showed how mathematics, more than most other branches, could be a constant source of inspiration under any conditions, because neither laboratories nor immense libraries are essential in the search after her truths. He advised that teachers set apart a definite amount of time each week, even if it be only a few hours, and devote it to quiet and earnest consideration of some mathematical subject. After suggesting some topics for such study, he added that reviews of previous work are often exceedingly profitable. This address was full of inspiration and encouragement for every teacher.

Professor H. W. Marsh, of Pratt Institute, Brooklyn, gave a brief resumé of the educational ideals of the past and then showed how the modern ideal of educating each child so as to render him of greatest service to society is a transition in *name only* from the cultural ideal so long predominant. The main effect of the cultural plan has been to drive a very large percentage of our children from school by the end of the fifth grade. Many attempts have been made to introduce problems of real life into mathematics, but these problems have been largely of an artificial type. The solution of this question can only be brought about by pre-

senting the fundamental propositions and principles of mathematics in such constructive and developmental form as to *make each student feel the joy of individual discovery and creation.* The address contained some specific suggestions as to the subject matter and method of presentation of mathematics to meet these new ideals.

Professor Stager emphasized the need of greater coöperation between the university and the secondary school. This can be shown by a sympathetic interest on the part of the former in the varied, and often difficult, problems of the latter. High standards should be set by the university and should be insisted upon. Helpful suggestions should be offered through visitation of university instructors to the schools and teachers' meetings, and through carefully prepared bulletins. The university entrance requirements should receive re-adjustment: (1) by defining the two years of required mathematics as *elementary mathematics,* rather than as *algebra and plane geometry,*[1] which would permit of greater efficiency in teaching and afford those teachers favorable to the idea of so-called "unified mathematics" an opportunity to work out their ideas; (2) by allowing the second year of algebra to be devoted, one half to the more important, but less theoretical portions, of the present requirement, and the other half to the elementary fundamental principles of analytical geometry, more especially of the straight line and the circle. This would prove equally advantageous to both the university and the secondary school.

The two sessions were each attended by about seventy-five teachers and the papers were followed by spirited discussions.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

Elementary Mathematical Analysis. A text-book for first year college students. By CHARLES S. SLICHTER. McGraw-Hill Book Company, New York, 1914. xiv + 490 pages.

Note.—The editors asked two men representing quite different phases of mathematical interests to review this book, and they take pleasure in presenting to the readers of the MONTHLY these two reviews.

I.

For years it has been assumed that mathematics should be administered in definite doses. After a pupil has emerged from the grammar school he is to be treated to one year of elementary algebra, one of plane geometry, then a half year each, of solid geometry, plane trigonometry, intermediate algebra, and advanced algebra. The tendency has been to prescribe the table of contents of each of these subjects. A student who can pass examinations in them all is ready to commence the study of analytic geometry, for which considerable familiarity

[1] This provision was made by the University of Chicago two years ago. EDITOR.

with graphical methods can be presupposed from the algebra. But in many cases the preparation in algebra has been found unsatisfactory, so that now attempts are being made to accept alternate subjects for college entrance and to provide for instruction in advanced algebra in college. In the book under review a much greater step is taken; only elementary algebra and plane geometry are presupposed; the volume attempts to provide instruction in trigonometry, intermediate and advanced algebra, and in plane analytic geometry. Use is made of solid geometry but once (p. 416).

To make such a programme economical, the school algebra and geometry should be given during the last two years of high-school rather than during the first two years as at present, otherwise the student would commence his college mathematics in a very rusty condition. Partial provision is made for this state of affairs, however, by including a sixteen page review of elementary algebra in the volume.

The table of contents gives only the names of the chapters, some of which contain several pages of material not connected with the title. This lack is partly compensated for by a full index. Several hundred exercises for the student are included, the answers to which are not furnished.

The first and second chapters are largely descriptive; they deal with the ideas of scale, functionality, and of graphical representation, with particular attention to power functions. The presentation is enlivened by a large number of illustrations of practical applications. The next chapter is a successful combination of the elements of trigonometry and analytic geometry of the circle. The trigonometric functions are at once defined for the general angle, thus making unnecessary much of the explanatory matter found in the ordinary texts. Indeed, this scheme could have been applied to a number of particular cases, where the author prefers to employ the ordinary right-angle triangle. It is to be regretted that the same generality was not always preserved in the later chapter on the addition formulas.

It seems odd to find in the chapter entitled "Ellipse and Hyperbola" the graphs of $y = \tan x$, $y = \cot x$, $y = \sec x$, and proofs of the theorems: limit $x/\sin x = 1$, limit $x/\tan x = 1$. The chapter on simultaneous equations gives the factor theorem and a number of graphical illustrations, but nothing more from the theory of equations. Synthetic division and Horner's method are no where mentioned.

Next follows a chapter on permutations and combinations, the main use of which is to prove the binomial theorem; the next chapter, on progressions, is provided with a number of graphical illustrations. The substance of these three chapters is usually included in elementary algebra.

The longest chapter (68 pages) and perhaps the best chapter is entitled "The logarithmic and trigonometric functions." It begins with a historical outline and accompanies it with graphical illustrations, making the development natural and interesting. After the elementary properties have been established, they are applied to explain the double scale, the slide rule, uses of logarithmic paper and the interpretation of data gathered from a number of engineering problems.

Chapter IX includes the machinery of trigonometry, the solution of triangles, the graph of a power of $\sin x$, etc.

Another commendable chapter is the tenth, on waves; it discusses the meaning of the terms used and shows how to apply the analysis in a large variety of ways. The chapter on complex numbers seems unnecessarily long (40 pages). It seems to the reviewer that all the essential facts could be stated in much less space.

The chapter on loci is concerned almost entirely with problems. They include several on the lemniscate, the cycloidal curves, and various others. The chapter on the conic sections contains the traditional treatment of many of the problems of analytic geometry, including tangents and normals, the general equation of a conic, and confocal conics. The criterion given for determining the positive and negative sides of a line is not applicable to lines which pass through the origin.

No mention is made of determinants. The idea of a derivative is introduced to obtain the slope of a general parabola, and of the exponential curve. The first case is easily justified, but the second is questionable, at first sight.

Whether this programme is a wise one can be determined only from experience. A priori praise or blame has but little meaning; at any rate it is an earnest effort to remove some of the difficulties of our present schedules.

VIRGIL SNYDER.

CORNELL UNIVERSITY.

II.

The first and most permanent impression which Slichter's Elementary Analysis makes upon one is of originality, of extraordinary and all-pervasive originality. The work is so different from any I have taught or any I know that I should be unwilling to give any estimate of its usability as a class text. Better than a review would be a series of statements by a number of persons who had used the book. And in this connection I might suggest that the MONTHLY could do no greater service to collegiate teaching and teachers of mathematics than by replacing or supplementing its "Book Reviews" by conducting an experience meeting on texts which have had a fairly wide adoption.

When we review an ordinary book we have only to compare it with others of its like and to make some rather inaccurate estimate of how it would impress a class. When we have to deal with an iconoclastic work we must go back to first principles, and these principles have not yet been definitely established, except in so far as the historical development of mathematical teaching to the present time may be considered to have established them, and this test is always unfair to the original work.

We shall therefore ask first the question: What should be taught in the freshman year? The canonical answer is, algebra, trigonometry, and analytic geometry, in various proportions. Now analytic geometry, except for that part which introduces the student to coördinates and elementary curve tracing and which may better be designated as graphical algebra, is a purely mathematical

subject which is distinctly difficult for a student to assimilate in his most immature collegiate year. The real problem of analytic geometry is to obtain geometrical results by algebraic means, and experience seems to show that this is too much for the freshman; moreover, it is beyond the needs of any but the student of pure mathematics or of really advanced applications. There has sprung up a strong tendency toward the elimination of true analytics from freshman work, and we hope the tendency will become a fixed habit. Slichter pays scant attention to analytics.

The process of attrition has also been applied to trigonometry. There are many old texts built upon the practice of nearly, if not quite, a whole year's work devoted to the subject. Undoubtedly plenty of material of importance exists from which a long course in trigonometry can be constructed, but not without calling upon many items which are unnecessary for the ordinary student. We are therefore getting down to twenty or thirty lessons on trigonometry, and Slichter has no more than this (at least of the canonical sort of trigonometry).

When our students come to college they have been over a considerable amount of algebra, probably as much as, or more than, most of them will need; but they frequently do not know how to use rapidly and accurately even the smaller amount which they must meet almost daily. A review of secondary school algebra would be a considerable portion of the first few weeks work in the freshman year if we put our students' efficiency before our academic pride. Slichter closes his book with such a review and with the statement that the review should precede Chapter I. (Why not print it there?) He does not include simultaneous quadratic equations in this review; they are treated appropriately but very inadequately, especially as regards exercises for the student to do, after the common second order curves have been presented. As for additional algebra, the author gives us permutations and combinations, the binomial theorem, and progressions, interesting material briefly treated.

We see therefore that as regards algebra, trigonometry, and analytic geometry we are on safe modern ground in retaining the essentials and in eliminating the more advanced parts.

The treatment of these three subjects is according to the mixed method. We do not proceed with one until it is completed but oscillate from one to another. The first two chapters are analytics (used, as always in this review, for graphical algebra or curve tracing), the third is trigonometry, the fourth returns to curve tracing and the fifth to its more algebraic side (solution of equations). Chapters VI and VII are algebraic, Chapter VIII is on logarithms and exponentials, and Chapter IX is on trigonometry again. The tenth and eleventh chapters on waves and complex numbers (with polar diagrams of periodic functions) are for the most part curve tracing once more, while the last two chapters on loci and the conic sections approach true analytics without quite reaching it.

It will be noticed that the author has really welded his material together; he does not switch violently from one subject to another just for the sake of pursuing the mixed method; there are modulations. Ordinarily it would not

be possible for the teacher to transpose the chapters and rearrange the work according to subjects—though I am by no means certain that such a transposition could not be made with Chapter IX so that it might follow Chapter III.

It is now time for a second fundamental question: Is the mixed method good? We all know the difficulty it is supposed to avoid, namely, the isolation of the student's knowledge into mutually insulated bodies. Some also feel that by not classifying a student's knowledge you give him greater power, you lead him to apply all his mathematical equipment to a new problem, instead of some particular part of his equipment such as trigonometry. The mixed method as contrasted with the compartment method of instruction is so young that we cannot yet say whether it is better or worse. There is the possibility, which at times appears to me as a strong probability, that by introducing broad connecting and conducting links between the different compartments you so reduce the potential that the efficiency is lowered. May it not be true that most problems which the majority of us solve are really problems in algebra, in trigonometry, or in analytics, and that the training in classification according to these topics is highly valuable? Is not the mixed method mixing to the student, and may not the oscillation be allied to vacillation? Only experience can answer.

As the author has reduced the ordinary subjects to such moderate compass, what has he done with the rest of his five hundred pages? He has introduced a large number of topics not usually found in a freshman text, perhaps not in any mathematical text used by mathematicians—double and triple scales, the slide-rule, plotting on logarithmic and semi-logarithmic and sinusoidal coördinate paper, simple harmonic motion, damped vibrations, logarithmic decrement, waves, shearing motion, graphs of many engineering formulas, and plenty of odds and ends. In these and in many other ways the work suggests Perry's "Calculus for Engineers."

It is true that all the things which the author takes up may logically be taken up; but we are brought back to the first question: What ought to go into the freshman course? For myself I do not believe in a mathematical or engineering museum for first year students. When a book starts in to give a series of illustrations and appends a footnote (as on page 68) stating: "The instructor is expected fully to explain the meaning of the technical terms here used," I get out of sorts with the author. The freshman is an immature person with a limited range of ideas and with a great many ideas in this limited range in need of clarification; the fewer unfamiliar terms that are introduced in his instruction the better, and the greater the chance of impressing on him the main thing in mathematical instruction, to wit, to think clearly.

Among the ideas which every freshman has, but has only vaguely, is the idea of speed. The word speed does not occur in Slichter's index, nor does velocity except for (uniform) angular velocity and velocity of a wave. The word rate is also missing. One may remark that without entering upon the notions of calculus these words cannot be defined. Precisely, and that is the fundamental reason for entering as promptly as possible upon those ideas. The author urges

that one of his chief aims is to impress the concept of functionality on the student. He does impress it so far as graphical algebra alone can impress it; but of prime importance in the idea of functionality is that of a rate of change, and into this he does not go except for the mention once or twice of slope.

Here is the really immoral element in the book: The author has cut down the algebra, the trigonometry, and the analytics to a point where the student has not acquired a real facility of operation with any of the three, and then instead of turning the time and space thus saved into a presentation of the elements of the calculus, he has frittered both away with frills which are more or less unintelligible to the student and of incomparably less value to him if understood.

There are three objects in mathematical instruction: (1) To give the student a training in formal manipulation which will suffice him for his ordinary needs—and to give this it is necessary to take him over considerably more than what he will actually use, just as to play satisfactorily a sonata one needs a technique developed beyond that of the piece; one must have reserve power. (2) To formulate for the student and to drill into him the fundamental ideas which he meets daily and with which in many cases he is already vaguely familiar—speed, rates, limits of sums—and here again he should go far enough to have reserve power, whereas most texts on calculus do not take him far enough so that he can formulate a definition for flux or induction. (3) To give him a taste of the applications of his ideas and his technique in mechanics or physics—and this could be done by the departments of mechanics and physics if they were not so terribly chary of using the mathematics that the student has (possibly the teacher has it no longer?).

Now it is axiomatic that you cannot teach all these subjects in a single year, and to freshmen at that. In particular (3) must be almost entirely omitted at first. You cannot even give a freshman both his technique and his ideas. In this dilemma the preference used to be given to technique, and with a good reason; for the young will unhesitatingly accept a formal method and, being imitative rather than rational, will rapidly acquire all the necessary facility. If this preference were still to be followed, we should give the freshman a thorough course in trigonometry, including trigonometric equations and identities, which are insufficiently treated by our author, and perhaps a few applications to statics in the plane; we should follow with a small amount of curve tracing, and then finish the formal differential calculus including a few geometric and kinematic applications. The sophomore year would begin with formal integration.

The serious objection of a practical nature to this procedure is that the ideas of the calculus do not come early enough to be most useful in the work in physics. We ought therefore to turn to the other preference and give the freshman a brief course in trigonometry (which might better be given in the secondary schools), a little curve tracing, and the ideas of the differential and integral calculus, keeping the formal work so simple that it will offer no difficulty. The sophomore year would then begin with trigonometric identities and drill in differentiation and integration. The technique will not come too late for the work in physics

and engineering, because not much technique is required for the early instruction in these subjects.

We have gone into these matters in such detail because we have had to go back to fundamentals and because we believe that the author has made the worst possible use of his extra space and time. We shall now try to give some justification for our feeling that the book is intensely original. The ellipse is defined as the locus obtained by shortening the ordinates of a circle in a definite ratio, and some two hundred pages later the tangent to the ellipse is found by applying the same rule to the ordinates of the tangent to the circle. The ellipse is given in parametric form, the loci $y = \tan x$ and $y = \sec x$ are plotted, and the hyperbola is defined and plotted from its parametric form $x = a \sec \theta$, $y = b \tan \theta$. The tangent to the hyperbola is not found. The tangent to the parabola is found by expressing the fact that it cuts the curve in coincident points; that to the circle, by its perpendicularity to the radius. The normal equation of a straight line and the distance from a point to a line occur near the end of the book in the chapter on conics, presumably because by means of them the passage from the tangent to a circle to the tangent to an ellipse is made.

At the very beginning of the book the student is told what drawing materials he should have and a considerable amount of graphical computation is given. This comes in handy in treating logarithms which are introduced by arranging arithmetic and geometric progressions in parallel columns. Exponential curves are constructed and the number e is defined as that value of r for which $y = r^x$ passes through (0, 1) with the slope 1. In this connection and only here is the slope related to the limit of the slope of a secant. Later e is found by a familiar operation on the binomial expansion, an operation which is justified by the (presumably untrue) statement that in the calculus it is shown that

$$\lim_{n=\infty}\left[1+1+\frac{1-\frac{1}{n}}{2!}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!}+\cdots\right]=1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots.$$

These illustrations, taken with earlier mention of unusual things discussed by the author, may indicate the originality of the work, an originality which grows on one constantly. Indeed it would be little exaggeration to say that if there was an unusual way of doing anything, the author has chosen that way and worked it down so smoothly that few who did not know would consider it new.

For one who lays such stress on calculation and gives such elaborate tables illustrating how an expert computer would arrange the solution of a triangle, it is surprising to find the author paying no attention to the vertical alignment of his decimal points in the table. I have not seen many examples of the work of expert computers, but what few I have seen look very different in this particular. Moreover, in solving a triangle with two sides and the included angle given, why use the tangent formulas? Why not be original and drop the perpendicular from the shorter side upon the longer? This solution may not be quite so neat, but is good enough, and saves the student the task of learning a new method.

The author is constantly and terribly mixed up in his statements and notations, though of course not in his ideas, about number and magnitude. Unless the reader can supply a great deal he cannot properly interpret the statements on pages 9, 43, 66, 343, and I fear that there are exercises (for the freshman) on this last page which I could not myself answer with any assurance of agreeing with the author.

Slichter's Elementary Mathematical Analysis should be widely tried out, if only for the rest that it will give the teacher from the familiar beaten paths; there is a charming freshness about the work and, whether we like it or not, it is bound to be ranked as a distinct contribution to the theory and practice of freshman instruction in mathematics.

E. B. WILSON.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

PROBLEMS FOR SOLUTION.

ALGEBRA.

When this issue was made up, no solutions had been received for numbers 417–426.

426. Proposed by HERBERT N. CARLETON, West Newbury, Mass.

Find all the solutions of the equation $x\sqrt[x]{x} = x^x$.

427. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

If $r \sin(\theta + \alpha) = m$ and $r \cos(\theta + \beta) = n$, show that

$$r = \frac{\sqrt{m^2 + n^2 - 2mn \sin(\alpha - \beta)}}{\cos(\alpha - \beta)},$$

GEOMETRY.

When this issue was made up, no solutions had been received for numbers 447–8, 450–454.

455. Proposed by R. P. BAKER, University of Iowa.

Find the minimum triangle of assigned angles inscribed in a given triangle.

456. Proposed by J. W. CLAWSON, Ursinus College.

The interior and exterior bisectors of the angles A, B, C of a triangle meet the opposite sides in U, U'; V, V'; W, W' respectively. Circles are drawn on UU', VV', WW' as diameters (Circles of Apollonius.) Prove that (1) These three circles have a common chord. (2) The centre of the circumcircle lies on this common chord.

CALCULUS.

When this issue was made up, no solutions had been received for numbers 358, 361–2, 364–372, 374–5, and 377.

376. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that

$$\frac{1}{z} - \frac{1}{z}(1 - 2xz + z^2)^{\frac{1}{2}} = x + \frac{z}{2}\left(\frac{x^2 - 1}{1 - xz}\right) + \sum_{n=2}^{\infty} \frac{1, 3, 5 \cdots 2n - 3}{2, 4, 6 \cdots 2n} (x^2 - 1)^n \left(\frac{z}{1 - xz}\right)^{2n-1}.$$

377. Proposed by W. D. CAIRNS, Oberlin College.

It is required to find a curve of the form $y = x(x - a)(x - b)$ such that the abscissas of the maximum and minimum values, as well as a and b, shall be positive integers.

MECHANICS.

When this issue was made up, no solutions had been received for numbers 289, 292–3, 295–299, 301, and 303.

302. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

A ball is projected from a given point at a given inclination β towards a vertical wall; determine the velocity so that after striking the wall the ball may return to the point of projection.

NUMBER THEORY.

When this issue was made up, no solutions bad been received for numbers 215–16, 218, 220, and 223–226.

226. Proposed by ELBERT H. CLARKE, Purdue University.

If 0! is taken equal to 1, and if k is any positive integer greater than or equal to 2, show that

$$\sum_{n=0}^{\infty} \frac{n!}{(k+n)!} = \frac{1}{(k-1)} \cdot \frac{1}{(k-1)!}.$$

227. Proposed by R. P. BAKER, University of Iowa.

Show that every rational number can be expressed as a finite sum $\sum_{n=m}^{n=m+k} \frac{a_n}{n}$, where a_n is either 0 or 1 and m is any positive integer.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

409. Proposed by C. E. GITHENS, Wheeling, W. Va.

Find integral values for the edges of a rectangular parallelopiped so that its diagonal shall be rational.

II. Solution by Artemas Martin, Washington, D. C.

On pp. 269–273 of the October, 1914, Monthly, W. C. Eells has solved a different problem from the one proposed. In making $x^2 + y^2 = \square$, he adds another condition not required.

Let x, y, z be the edges and d the diagonal of the parallelepiped; then we have to satisfy the equation

$$x^2 + y^2 + z^2 = d^2.$$

It is not necessary that $x^2 + y^2$ be a square. Let us assume $x = a$, $y = b$, $z + c = d$, and we have

$$a^2 + b^2 + z^2 = (z + c)^2 = z^2 + 2cz + c^2,$$

which immediately gives

$$z = \frac{a^2 + b^2 - c^2}{2c} \qquad \text{and} \qquad d = \frac{a^2 + b^2 + c^2}{2c}.$$

Therefore

$$a^2+b^2+\left(\frac{a^2+b^2-c^2}{2c}\right)^2=\left(\frac{a^2+b^2+c^2}{2c}\right)^2,$$

whatever be the values of a, b, c.

1. Taking $a=1$, $b=2$, $c=1$, we find

$$1^2+2^2+2^2=3^2,$$

which is the *smallest* rational parallelepiped.

2. Taking $a=2$, $b=3$, $c=1$, we have

$$2^2+3^2+6^2=7^2,$$

which is the smallest rational parallelepiped having its edges all different.

3. Taking $a=1$, $b=4$, $c=1$, we get

$$1^2+4^2+8^2=9^2.$$

4. Let $a=2$, $b=6$, $c=2$, and we have

$$2^2+6^2+9^2=11^2.$$

5. If $a=3$, $b=4$, $c=1$, we get

$$3^2+4^2+12^2=13^2,$$

which, on p. 269, is stated to be "the smallest rational parallelopiped."

6. If we take $a=8$, $b=9$, $c=5$, we will get

$$8^2+9^2+12^2=17^2.$$

7. Taking $a=4$, $b=8$, $c=2$, we have

$$4^2+8^2+19^2=21^2.$$

8. If $a=3$, $b=4$, $c=3$, then we get

$$12^2+15^2+16^2=25^2.$$

And so on, there being an infinite number of parallelopipeds whose edges and solid diagonals are rational integers.

The condition $x^2+y^2+z^2=\square$ can be satisfied in many ways.—See *Mathematical Magazine*, Vol. II., No. 5 (October, 1891), pp. 71–74. Other methods are given in a forthcoming paper which will appear in the third part of No. 12, Vol. II., of the *Mathematical Magazine*.

416A. Proposed by H. O. HANSON, East Elmhurst, N. Y.

Find the nth term and the sum of n terms of the series obeying the relation $u_i=u_{i-1}+2u_{i-2}$ in terms of n and the first two terms, u_1 and u_2, these two terms being arbitrary.

This problem was incorrectly numbered 416.

SOLUTION BY S. A. JOFFE, New York City.

Adding u_{i-1} to both members of the given relation, we obtain the following recurring formula:

$$u_i + u_{i-1} = 2(u_{i-1} + u_{i-2}). \tag{1}$$

If we replace i successively by n, $n-1$, $n-2$, $\cdots$, 3 and multiply the resulting $n-2$ equations, we have, after cancellation:

$$u_n + u_{n-1} = 2^{n-2}(u_2 + u_1). \tag{2}$$

Similarly

$$u_{n-2} + u_{n-3} = 2^{n-4}(u_2 + u_1),$$

$$u_{n-4} + u_{n-5} = 2^{n-6}(u_2 + u_1),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot,$$

the last equation of this type being $u_2 + u_1 = u_2 + u_1$, or $u_3 + u_2 = 2(u_2 + u_1)$, according as n is even or odd.

Adding all these equations, we have:

for n = even, $\sum_{i=1}^{n} u_i = (2^{n-2} + 2^{n-4} + 2^{n-6} + \cdots + 1)(u_2 + u_1)$,

or

$$\sum_{i=1}^{n} u_i = \tfrac{1}{3}(2^n - 1)(u_2 + u_1); \tag{3}$$

and for n = odd,

$$\sum_{i=1}^{n} u_i = (2^{n-2} + 2^{n-4} + 2^{n-6} + \cdots + 2)(u_2 + u_1) + u_1 = \tfrac{2}{3}(2^{n-1} - 1)(u_2 + u_1) + u_1,$$

or

$$\sum_{i=1}^{n} u_i = \tfrac{1}{3}(2^n - 2)(u_2 + u_1) + u_1. \tag{4}$$

Formulæ (3) and (4) express the sum of the first n terms, as required, in terms of n and the first two terms; the nth term u_n is found by subtracting $\sum_{i=1}^{n-1} u_i$ from $\sum_{i=1}^{n} u_i$, the result being for n = even,

$$u_n = \tfrac{1}{3}(2^n - 1)(u_2 + u_1) - [\tfrac{1}{3}(2^{n-1} - 2)(u_2 + u_1) + u_1]$$

$$= \tfrac{1}{3}(2^{n-1} + 1)(u_2 + u_1) - u_1;$$

and for n = odd, $u_n = [\tfrac{1}{3}(2^n - 2)(u_2 + u_1) + u_1] - \tfrac{1}{3}(2^{n-1} - 1)(u_2 + u_1)$

$$= \tfrac{1}{3}(2^{n-1} - 1)(u_2 + u_1) + u_1.$$

The last two formulæ may be combined into one, for n in general,

$$u_n = \tfrac{1}{3}[2^{n-1} + (-1)^n](u_2 + u_1) + (-1)^{n-1}u_1. \tag{5}$$

Also solved by A. M. HARDING, A. H. HOLMES, HORACE OLSON, and the PROPOSER.

417A. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Solve $E(x^2) - E(3x) = 7$, where $E(m)$ is the largest integer in m. The value of x is to be in the form $4 + (y/32)$ where y is an integer.

This problem was incorrectly numbered 417.

SOLUTION BY A. M. HARDING, University of Arkansas.

If we draw the graphs of the functions $E(x^2)$ and $E(3x) + 7$ we find

$$E(x^2) = E(3x) + 7 = 20, \quad \text{if} \quad \sqrt{20} < x < \sqrt{21}; \quad \text{i. e.} \quad 4.4721 < x < 4.5826;$$

$$E(x^2) = E(3x) + 7 = 21, \quad \text{if} \quad 14/3 < x < \sqrt{22}; \quad \text{i. e.} \quad 4.6667 < x < 4.6904;$$

$$E(x^2) = E(3x) + 7 = 2, \quad \text{if} \quad -5/3 < x < -\sqrt{2},$$

$$\text{i. e.} \quad -1.6667 < x < -1.4142.$$

Now we must have $x = 4 + y/32$ where y is an integer. Hence

$$4.4721 < 4 + y/32 < 4.5826, \qquad \therefore y = 16, 17, \text{ or } 18.$$

And

$$4.6667 < 4 + y/32 < 4.6904, \qquad \therefore y = 22.$$

$$-1.6667 < 4 + y/32 < -1.4142, \qquad \therefore y = -181, -180, \cdots, -174.$$

GEOMETRY.

430. Proposed by DANIEL KRETH, Wellman, Iowa.

The distance between A and B is always a feet. A travels along a straight path at the rate of v_1 miles per hour, and B starts at the same time in the path behind A and travels in a curve at the rate of v_2 miles per hour. How far will B travel to reach the path in front of A, and how far to reach the path again behind A?

SOLUTION BY ELIJAH SWIFT, University of Vermont.

Take the origin at the place where B starts and the Y-axis through A. Then the coördinates of A at any time, t, are $(0, a + v_1t)$, and those of B must satisfy the equations

$$(1) \quad x^2 + (y - a - v_1t)^2 = a,^2 \qquad (2) \quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = v_2^2.$$

Solving (1) for y, we have $y = -\sqrt{a^2 - x^2} + a + v_1t$, where the negative sign holds until B is abreast of A.

Substituting this value in (2), we have a quadratic equation in dx/dt, from which we find

$$\frac{dx}{dt} = \frac{\sqrt{a^2 - x^2}}{a^2}\{-v_1x + \sqrt{v_1^2x^2 - a^2v_1^2 + a^2v_2^2}\},$$

the plus sign of the second radical being taken since dx/dt is positive.

From equation (2), we deduce $ds = v_2dt$. Hence, we have

$$s \equiv v_2t = 2 \cdot v_2 \cdot a^2 \int_0^a \frac{dx}{\sqrt{a^2 - x^2}\{-v_1x + \sqrt{v_1^2x^2 - a^2v_1^2 + a^2v_2^2}\}},$$

as the total distance is evidently twice as far as the distance B must cover to be abreast of A. This would seem to be an elliptic integral and hence not easily evaluated.

The same reasoning would hold on the return, except for signs, and we obtain the same expression for s except for the sign of $v_1 x$, which would be plus.

Note.—This problem should evidently have been listed under Calculus.

432. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Having given a tetrahedron, a, b, c, d, e, f, find an expression for the radius of the sphere which is tangent to the six edges.

SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

First, a solution is not possible in the general case. Eight spheres can be drawn to touch any four of the six edges, but none of them will touch the other two edges unless certain conditions are satisfied.

If the tetrahedron is of such a shape that a sphere can be *inscribed* to touch all the edges, the planes of the four faces of the tetrahedron cut the sphere in circles inscribed to the triangles which form the faces. At each point where one of these four circles touches an edge, another of the circles also touches it. Now if T is the point where the in-circle of the triangle ABC, whose edges are a, b, c touches the side AC, then $CT = (a + b - c)/2$; and if T is the point where the in-circle of the triangle ACD whose edges are f, d, b touches the side AC, then $CT = (b + f - d)/2$.

Therefore $a + d = c + f$.

Similarly it can be shown that $a + d = b + e$.

Hence, a sphere can be drawn to touch the six edges internally if

$$a + d = b + e = c + f \tag{1}$$

Similarly it can be shown that a sphere can be drawn tangent to the six edges, touching b, c, d *produced*, if $a - d = e - b = f - c$; that a sphere can be drawn tangent to the six edges, touching a, b, c *produced* if $a - d = e - b = c - f$; one touching a, b, f, *produced* if $a - d = b - e = f - c$; one touching e, d, f *produced* if $a - d = b - e = c - f$.

Thus, if either

$$a + d = b + e = c + f,$$

or $a - d = b - e = c - f$, or $a - d = b - e = f - c$,

or $a - d = e - b = c - f$, or $a - d = e - b = f - c$,

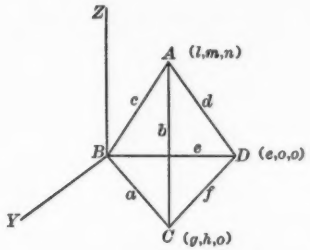
a sphere can be found to touch the six edges.

If any face is an equilateral triangle and the other three edges are equal, two such spheres can be drawn, one inscribed, the other escribed, touching the sides of the equilateral triangle and the three other edges produced.

If the opposite edges are equal in pairs (and this includes the case of the regular tetrahedron), the inscribed and four escribed spheres can be drawn.

Secondly, to find the radius of a sphere touching the edges a, c, e, f. Take BD for axis of X, BY perpendicular to BD in the plane of the triangle BCD, BZ perpendicular to BD and BY. Take C to be the point (g, h, o), A to be (l, m, n).

The centre of the sphere whose radius is required is equidistant from BD, BC, CD, AB.



Now the equations of the locus of points equidistant from BD and BC are

$$\begin{cases} hx - (a+g)y = 0, \\ hx + (a-g)y = 0, \end{cases}$$

the equations of the locus of points equidistant from CD and BD are

$$\begin{cases} hx + (e-g+f)y = eh, \\ hx + (e-g-f)y = eh, \end{cases}$$

and the equations of the locus of points equidistant from AB and BD are

$$\begin{cases} (c-l)x - my - nz = 0, \\ (c+l)x + my + nz = 0. \end{cases}$$

Taking the first of each pair of equations, we find that the center of the sphere *inscribed* to the tetrahedron touching these four edges is the point

$$\left(\frac{e(g+a)}{a+e+f}, \frac{eh}{a+e+f}, \frac{e[(c-l)(a+g) - hm]}{n(a+e+f)} \right).$$

The radius, R, of this sphere is given by the equation:

$$R = \frac{e}{n(a+e+f)} \sqrt{h^2n^2 + [(c-l)(a+g) - hm]^2}. \tag{2}$$

We must now eliminate g, h, l, m, n from (2), making use of the relations $l^2 + m^2 + n^2 = c^2$, $g^2 + h^2 = a^2$, $(e-g)^2 + h^2 = f^2$, $(l-e)^2 + m^2 + n^2 = d^2$, $(l-g)^2 + (m-h)^2 + n^2 = b^2$.

From these five equations can be obtained the following five: $g = (a^2 + e^2 - f^2)/2e$, $l = (c^2 + e^2 - d^2)/2e$, $gl + hm = (c^2 + a^2 - b^2)/2$, $h^2 = (2a^2e^2 + 2e^2f^2 + 2f^2a^2 - a^4 - e^4 - f^4)/4e^2$, $n^2 = [a^2d^2(e^2 + f^2 - a^2) + b^2e^2(f^2 + a^2 - e^2) + c^2f^2(a^2 + e^2 - f^2) - a^2(d^2 - b^2)(d^2 - c^2) + c^2(b^2 - e^2)(b^2 - d^2) - f^2(c^2 - d^2)(c^2 + b^2) - a^2e^2f^2] \div [2a^2e^2 + 2e^2f^2 + 2a^2f^2 - a^4 - e^4 - f^4]$.

Substituting these values in (2), we obtain,

$$R = \frac{1}{2n} \frac{1}{a+e+f} \sqrt{[(a+e)^2 - f^2][(e-c)^2 - d^2][(c-a)^2 - b^2]}.$$

Now the volume of the tetrahedron is $V = ehn/6$.

Using this fact, a little more reduction leads to the result

$$R = \frac{(a+e-f)}{24V}\sqrt{[(e-c)^2-d^2][(c-a)^2-b^2][f^2-(a-e)^2]}, \tag{3}$$

where

$$\begin{aligned}144V^2 = {} & a^2d^2(e^2+f^2-a^2)+b^2e^2(f^2+a^2-e^2)+c^2f^2(a^2+e^2-f^2)\\ & -a^2(d^2-b^2)(d^2-c^2)-e^2(b^2-c^2)(b^2-d^2)-f^2(c^2-d^2)(c^2-b^2)-a^2e^2f^2.\end{aligned} \tag{4}$$

Seven similar expressions can be found for the radii of spheres tangent to the four edges a, c, e, f, touching one or more of the edges produced beyond the tetrahedron.

Thirdly, applying the conditions (1) to (3), we can obtain the required answer in a symmetrical form. Now (3) is the radius of a sphere touching the four edges a, c, e, f. If this sphere touches also the edges b, d, that is, if $a+d=b+e=c+f$, then

$$R = \frac{1}{24V}\sqrt[3]{\begin{array}{l}(a+b-c)(a-b+c)(-a+b+c)(b+d-f)\\ (b-d+f)(-b+d+f)(c+d-e)(c-d+e)\\ (-c+d+e)(a+e-f)(a-e+f)(-a+e+f),\end{array}} \tag{5}$$

where V has the value given in (4), which is already expressed in a symmetrical form. This is the required solution.

It may be added that, if A_1, A_2, A_3, A_4 are the areas of the four face-triangles, and r_1, r_2, r_3, r_4 the radii of the circles inscribed to these triangles, the above expression takes the simple form

$$R = \frac{2}{3V}\sqrt[3]{A_1A_2A_3A_4r_1r_2r_3r_4}. \tag{6}$$

If the tetrahedron is regular the expression reduces to

$$R = \frac{a}{2\sqrt{2}}.$$

Similarly the radii of the four spheres R_1, R_2, R_3, R_4 which touch three edges and the other three edges produced, are

$$R_1 = \frac{2}{3V}\sqrt[3]{A_1A_2A_3A_4r_1r_{21}r_{31}r_{41}}; \qquad R_2 = \frac{2}{3V}\sqrt[3]{A_1A_2A_3A_4r_2r_{12}r_{32}r_{42}};$$

$$R_3 = \frac{2}{3V}\sqrt[3]{A_1A_2A_3A_4r_3r_{13}r_{23}r_{43}}; \qquad R_4 = \frac{2}{3V}\sqrt[3]{A_1A_2A_3A_4r_4r_{14}r_{24}r_{34}},$$

where r_{pq} is the radius of the circle escribed to the triangle A_p which touches the edge common to A_p and A_q and the other two edges of A_p produced.

If the tetrahedron is regular, $R_1 = R_2 = R_3 = R_4 = \dfrac{3a}{2\sqrt{2}}$.

CALCULUS.

358. Proposed by C. N. SCHMALL, New York City.

About a given circle circumscribe the smallest parabola.

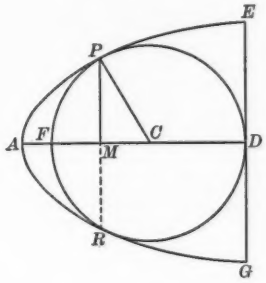
SOLUTION BY HORACE OLSON, Chicago, Illinois.

We assume that by the smallest parabola is meant the smallest segment of a parabola having its bounding ordinate tangent to the given circle.

Let $PFRD$ be the given circle whose center C is the origin of rectangular coördinates. Then, without loss of generality, the parabola may be assumed to have its axis on the axis of abscissas AD.

Let EAG be the required parabola. Then, if r is the radius of the circle, we have $x^2 + y^2 = r^2$ for the circle, and $y^2 = 2p(x + k)$ for the parabola, where p is the distance from the focus to the directrix and k is the distance AC.

Since P is the point of tangency, solving these two simultaneous equations subject to the condition of tangency, we have $r^2 - 2kp + p^2 = 0$.



Hence, $k = p/2 + r^2/2p$.
Then

$$AD = AC + CD = \frac{p}{2} + \frac{r^2}{2p} + r = \frac{(r+p)^2}{2p},$$

and

$$ED = \sqrt{2p(AD)} = (r + p).$$

Area of $EAG = a = \frac{2}{3}AD \cdot EG = \frac{4}{3}(r+p)^2/2p \cdot (r+p) = \frac{2}{3}(r+p)^3/p$.
Equating to zero the derivative of a with respect p, we have

$$\frac{da}{dp} = \frac{2}{3} \cdot \frac{3p(r+p)^2 - (r+p)^3}{p^2} = 0;$$

whence

$$(p + r)^2 = 0 \quad \text{and} \quad 2p - r = 0.$$

Hence, $p = -r$ or $p = \frac{1}{2}r$. The value $p = -r$ gives neither a maximum nor a minimum. The value $p = \frac{1}{2}r$ gives a minimum, and the equation of the corresponding parabola is $y^2 = r(x + 5r)/4$.

A similar solution was received from the PROPOSER.

359. Proposed by W. D. CAIRNS, Oberlin College.

Examine for maxima and minima

$$f(x) = e^{-cx}(1 + \cos x) \quad (c > 0)$$

SOLUTION BY A. M. HARDING, University of Arkansas.

$$f'(x) = -e^{-cx}(c + c\cos x + \sin x).$$

$$f''(x) = e^{-cx}(c^2 + c^2\cos x + 2c\sin x - \cos x).$$

Now $f'(x) = 0$ only when $c + c\cos x + \sin x = 0$, that is, when

$$c(1 + \cos x) + \sin x = 0,$$

or

$$2c\cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2} = 0.$$

Hence,

$$\cos\frac{x}{2} = 0, \quad \text{and} \quad x = \pi, 3\pi, 5\pi, \cdots (2n-1)\pi, \cdots$$

or

$$c\cos\frac{x}{2} + \sin\frac{x}{2} = 0, \quad \text{and} \quad x = 2\arctan(-c).$$

When $x = \pi, 3\pi, 5\pi, \cdots, (2n-1)\pi, \cdots, f''(x) = -c\,(2n-1)\pi$, which is always positive.

Hence, the minimum value of $f(x)$ is obtained by giving x any of these values. When $x = 2\arctan(-c)$, that is, $c\cos x/2 + \sin x/2 = 0$,

$$f''(x) = e^{-cx}[\{c^2(1+\cos x) + c\sin x\} + c\sin x - \cos x]$$

$$= e^{-cx}\left[2c\cos\frac{x}{2}\left(c\cos\frac{x}{2} + \sin\frac{x}{2}\right) + 2c\sin\frac{x}{2}\cos\frac{x}{2} - \cos x\right]$$

$$= e^{-cx}\left[2c\sin\frac{x}{2}\cos\frac{x}{2} - \cos x\right] = e^{-cx}\left[2\sin\frac{x}{2}\left(-\sin\frac{x}{2}\right) - \cos x\right]$$

$$= -e^{-cx}\left[2\sin^2\frac{x}{2} + \cos x\right] = -e^{-cx}[1 - \cos x + \cos x] = -e^{-cx}.$$

Now $-e^{-cx}$ is negative for the above value of x. Hence the maximum value of $f(x)$ is obtained by giving x the value $2\arctan(-c)$.

Also solved by W. C. EELLS, PAUL CAPRON, H. C. FEEMSTER, G. W. HARTWELL and the PROPOSER.

MECHANICS.

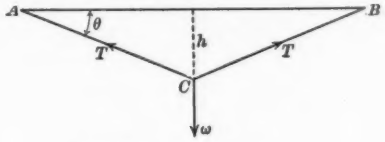
286. Proposed by C. N. SCHMALL, New York City.

A slightly elastic string is just long enough to reach between two hooks on the same horizontal line. A ring of weight w is placed at its middle point. Show that the ring will sink through a distance $h = a\sqrt[3]{3e\omega/2}$, where e is the elasticity of the string and $2a$ the distance between the two hooks.

SOLUTION BY B. F. FINKEL, Drury College.

Since w is at the middle point of the string, the tension T in the two halves of the string is the same when the string is in equilibrium. Let θ be the angle which the string makes with the horizontal line.

Then $AC = a \sec\theta$. Hence the strain in AC is



$$\frac{a\sec\theta - a}{a} = \sec\theta - 1.$$

By Hooke's Law, we have $T/(\sec\theta - 1) = e.$ (1)

For equilibrium, we have $2T\sin\theta = w.$ (2)

Hence,

$$2e\sin\theta(\sec\theta - 1) = w,$$

or

$$\tan\theta - \sin\theta = \frac{w}{2e}. \tag{3}$$

Since the string is only slightly elastic, θ must be small.

When θ is small we may take $\tan\theta = \theta + (\theta^3/3)$ and $\sin\theta = \theta - (\theta^3/6)$. Hence, substituting these values in (3) and solving for θ, we have $\theta = \sqrt[3]{w/e}$.

But $\tan\theta = h/a$, or $\theta + (\theta^3/3) = h/a$. Substituting for θ and solving for h, we have

$$h = a\left\{\sqrt[3]{\frac{w}{e}} + \frac{w}{3e}\right\}.$$

The proposer's result is incorrect, as the test of physical dimensions shows.

290. Proposed by B. F. FINKEL, Drury College.

A fox, pursued by a hound, is running with uniform velocity over a frail arch in the form of a cycloid: the hound stops at a weak point of the arch, then tumbles through, and reaches the level ground with a velocity equal to that of the fox. Prove that the fox exerted no normal pressure on the arch at the point where the hound fell through. Walton's *Problems in Theoretical Mechanics*, p. 662.

SOLUTION BY THE PROPOSER.

Let $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ be the parametric equations of the cycloidal arch and v_0, the speed of the fox. Then at the point where the hound tumbled through, we have the relation $v_0^2 = 2gy_0$, where y_0 is the ordinate of the arch at that point.

Now the centripetal force of the fox is $m(v_0^2/R)$, where m is his mass and R the radius of curvature of the arch. The force exerted by gravity in opposition to this force is $mg\cos\phi$, where ϕ is the direction of the arch at any point.

Hence, the normal pressure on the arch at any point, assuming $m = 1$, is

$$P = \frac{v_0^2}{R} - g\cos\phi.$$

From the equations of the arch, we have

$$\frac{dy}{dx} = \tan\phi = \cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right).$$

Whence, $\phi = (\pi/2) - (\theta/2)$. Also

$$R = \frac{[1 + (dy/dx)^2]^{\frac{3}{2}}}{d^2y/dx^2} = 4a \sin \theta/2$$

Hence, $P = v_0^2 \div (4a \sin \theta/2) - g \sin \theta/2$. But, since $y = \theta(1 - \cos \theta)$,

$$2 \sin^2 \frac{\theta}{2} = \frac{y}{a}.$$

Hence,

$$P = \frac{v_0^2 - 4ag \sin^2 \theta/2}{4a \sin \theta/2} = \frac{v_0^2 - 2gy}{4a \sin \theta/2},$$

But, at the point where the hound tumbled through, $y = y_0 = (v_0/2g)^2$. Hence, at that point, $P = 0$.

NUMBER THEORY.

212. Proposed by C. N. SCHMALL, New York City.

Given any positive integer N greater than 1; to prove that the sum of all the positive integers less than N and prime to N equals $\frac{1}{2}N \cdot \phi(N)$.

SOLUTION BY H. C. FEEMSTER, York College, Nebraska.

Let $N = a^h b^k c^l \cdots$, where $a, b, c, \cdots$ are primes. Then

$$\phi(N) = N \cdot \frac{a-1}{a} \cdot \frac{b-1}{b} \cdot \frac{c-1}{c} \cdot \cdots = N\left[1 - \Sigma\frac{1}{a} + \Sigma\frac{1}{ab} - \Sigma\frac{1}{abc} + \cdots\right],$$

the number of positive integers less than N and prime to N.

For every number $p < N/2$ and prime to N, there is a number $N - p > N/2$ and less than N and prime to N. Now there are $\phi(N)$ of these numbers or $\frac{1}{2}\phi(N)$ pairs of these numbers. But the sum of each pair is N, so the entire sum is $\frac{1}{2}N \cdot \phi(N)$, as required.

213. Proposed by R. D. CARMICHAEL, Indiana University.

Prove that no relatively prime integers x and y exist such that the difference of their fourth powers is a perfect cube.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

We are to prove the non-existence of any equation $x^4 - y^4 = z^3$. There are two cases to consider: (1) *x and y both odd*; (2) *one even and one odd.*

In the second case $x^2 + y^2$ and $x^2 - y^2$ are prime to each other, since any common factor would divide their sum $2x^2$, and difference $2y^2$; but these have, by hypothesis, 2 as their only common factor, and 2 is not a factor of $x^2 + y^2$ because x is even and y odd, or vice versa.

Since the product of $x^2 + y^2$ and $x^2 - y^2$ is a perfect cube, each of the factors must be, say, $x^2 + y^2 = a^3$, $x^2 - y^2 = b^3$. Call $2xy = c$. Then $a^6 - b^6 = c^2$, which is impossible. (See Number Theory, problem 209 in the March, 1914, MONTHLY which denies the existence of such an equation.)

Suppose now that x and y are both odd. Let $x = 4k \pm 1, y = 4l \pm 1$. Then $x^2 + y^2 = 16(k^2 + l^2) + 8(\pm k \pm l) + 2$, which is divisible by 2 but not by 4; and $x^2 - y^2 = 16(k^2 - l^2) + 8(\pm k \pm l)$, which is divisible by 8. Since $z^3 = (x^2 + y^2)(x^2 - y^2)$, it is divisible by 16 and hence by 64; and $x^2 - y^2$ by 32. Accordingly, we may write $x^2 + y^2 = 2\alpha^3$, $x^2 - y^2 = 32\beta^3$. We now see that if $x = 4k + 1$, $y = 4l + 1$, then $x + y$ is divisible by 2, but not by 4; any other combination of signs leads to this result for $x + y$ or $x - y$. We may then write, in the one case, $x + y = 2\gamma^3$, $x - y = 16\delta^3$. The other case may be similarly treated. Then

$$(x + y)^2 + (x - y)^2 = 4\gamma^6 + 256\delta^6 = 2x^2 + 2y^2 = 4\alpha^3.$$

Hence $\gamma^6 + 64\delta^6 = \alpha^3$, or $(\gamma^2)^3 + (2\delta^2)^3 = \alpha^3$, which is impossible. Hence there are no integral values of x and y *prime to each other* which satisfy the given equation.

Also solved by J. L. RILEY

QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL.

At the time of making up copy for this issue replies had not yet been received for numbers 4, 8, 11, 12, 13, 16, 18.

NEW QUESTIONS.

22. What can the Colleges do toward improving the teaching of mathematics in secondary schools?

23. What should be done with the theory of limits in elementary geometry? Should the recommendation of the National Committee of Fifteen on Geometry Syllabus be universally adopted? If not, what better disposition of the subject can be made?

REPLIES.

10. What use has been made of regular conference periods for assistance to individual students of secondary and college mathematics, and what services may they render?

REPLY BY CLYDE S. ATCHISON, Washington and Jefferson College.

For the past two years, one hour each day has been set aside for conference in mathematics, during which time one member of the department is in his class room ready to assist students in settling their individual difficulties. The privilege thus offered has been much appreciated by a majority of the students, of whom many seize the opportunity for extra instruction to enlarge their grasp of the subject, while others, of less ability or unfortunate preparatory training, make use of the occasion to have explained to them those points to which only a limited amount of time can be devoted in the regular class work. A frank announcement in the class room, that no time will be wasted with a man who does not keep up with the work of the course, and that a conference hour is not a time

when a man's lessons will be prepared for him, has forestalled any attempt on the part of students to take an unfair advantage of the hour.

The personal touch with the men, in a way which is not possible during the class period, together with what has been accomplished in the way of developing both the weaker students and those of more than average ability, has made these conference hours of inestimable value.

3. In connection with the theory of the conduction of electricity through gases, one is led to the differential equation

$$y\frac{d^2y}{dx^2} + a\left(\frac{dy}{dx}\right)^2 + b\frac{dy}{dx} + cy + d = 0, \tag{1}$$

where a, b, c, d are constants. For unrestricted values of a, b, c, d the solution of this differential equation presents peculiar difficulties, the series solutions obtained by the customary methods having (apparently) too small a range of convergence to be satisfactory from the point of view of electrical theory. The general solution of this equation is wanted in case it can be found, If no general solution is obtained for unrestricted a, b, c, d, it is desirable to know special values of a, b, c, d or special relations among a, b, c, d which make it possible to find the general solution; and this solution is desired in each case.

REPLY BY BARNEM LIBBY, University of Michigan.

The given equation may be written

$$yy'' + ay'^2 + by' + cy + d = 0. \tag{1}$$

Consider the equation

$$yy'' + ay'^2 + by' + cy = 0. \tag{2}$$

This is equivalent to

$$[ayy' + by]' + y[(1 - a)y' + cx]' = 0. \tag{3}$$

Hence (3) is exact if the coefficient of y is zero; and this is the case if $c = 0$ and $a = 1$.

We then get in (1)

$$[yy' + by]' = -d, \tag{4}$$

whence,

$$yy' + by = -d \cdot x + l, \tag{5}$$

l being a constant of integration.

To integrate

$$ydy + (by + d \cdot x - l)dx = 0, \tag{5'}$$

we put

$$\begin{cases} x = x_1 + \alpha, \\ y = y_1 + \beta. \end{cases}$$

Then $dx = dx_1$, $dy = dy_1$ and (5') becomes

$$(y_1 + \beta)dy_1 + (by_1 + d \cdot x_1 + b\beta + d \cdot \alpha - l)dx_1 = 0. \tag{6}$$

If now $\beta = 0$ and $d \cdot \alpha = l$, this becomes

$$y_1dy_1 + (by_1 + d \cdot x_1)dx_1 = 0. \tag{6'}$$

Let $y_1 = vx_1$, then $dy_1 = vdx_1 + x_1dv$, and then

$$vx_1(vdx_1 + x_1dv) + (bvx_1 + d \cdot x_1)dx_1 = 0,$$

or

$$v(vdx_1 + x_1dv) + (bv + d)dx_1 = 0, \tag{7}$$

i. e.,

$$(v^2 + bv + d)dx_1 + vx_1dv = 0,$$

or

$$\frac{vdv}{v^2 + bv + d} = -\frac{dx_1}{x_1}, \tag{7'}$$

or

$$\frac{\frac{1}{2}(2v + b) - \frac{1}{2}b}{v^2 + bv + d} = -\frac{dx_1}{x_1}.$$

Hence,

$$\log (v^2 + bx + d)x_1^2 - b\int \frac{dv}{v^2 + bv + d} = k. \tag{8}$$

Evaluating the indefinite integral and substituting the values of v^2 and v, we obtain a solution for y in terms of x.

NOTES AND NEWS.

EDITED BY W. D. CAIRNS.

Mr. JOHN BRANDEBERRY, A.B., Mt. Union College, 1914, and Miss MARVEL C. HORN, A.B., Ohio State University, 1914, have been elected graduate assistants in mathematics at Ohio State University, for the present academic year.

Mr. RAYMOND DU HADWAY, who has been studying at Göttingen the past year was forced to leave Germany on account of the war. He has taken a place to teach mathematics in Washington University, St. Louis, Mo.

Professors PIERRE BOUTROUX and J. H. M. WEDDERBURN, of the department of mathematics, Princeton University, have been granted leaves of absence; the former is enrolled in the French service, the latter in the British service.

At the University of Texas, Dr. DAVID F. BARROW has been appointed instructor in applied mathematics, and Mr. F. A. LA MOTTE instructor in pure mathematics. The new courses being given are: The calculus of variations, by Mr. ETTLINGER; and the mathematics of investment and life insurance, by Professor DODD.

It was proposed to hold the next meeting of the International Commission on the Teaching of Mathematics at Munich, Germany, August 2–5, 1915; but this meeting has been indefinitely postponed on account of the war. The main sub-

ject for discussion was to have been the preparation, theoretic and practical, of teachers of mathematics for the various grades. At the request of the Central Committee Professor Gino Loria, Genoa, Italy, has assumed general charge of the work relative to teachers of secondary mathematics.

According to the list of members published in the July, 1914, issue of the *Revista de la Sociedad Matemática Española*, the number of members of the Spanish Mathematical Society is now 436. Three of these members live in the United States.

The October, 1914, number of the *Journal of the Indian Mathematical Society* contains an article by PHILIP E. B. JOURDAIN entitled "The Theories of Irrational Numbers, Part I." According to the author's own words "the aim of this historical and critical study is somewhat different from that of the other works known to us which deal with the development of the theory of convergence and allied topics. We shall, in fact, be concerned primarily with questions of principle."

The National Academy of Sciences began in January, 1914, the publication of a monthly periodical called *Proceedings*. Professor E. H. MOORE, of the University of Chicago, is the mathematical member on the editorial staff. Professor E. B. WILSON, Massachusetts Institute of Technology, is the managing editor. The articles are expected to be short and to exhibit a summary of the most important results obtained by Americans in various fields of science.

Professor E. W. HOBSON gave six lectures at Cambridge University during the Easter term of last year on the history of the problem of the quadrature of the circle. These lectures have been published by the Cambridge University Press in a book entitled "Squaring the Circle. A History of the Problem." The price of the book is three shillings.

A London press dispatch says that "King George has approved the presentation by the council of The Royal Society of a royal gold medal to Professor ERNEST W. BROWN of Yale University for his investigations in astronomy."

WILLIAM FROTHINGHAM BRADBURY, author of textbooks on algebra, geometry, and trigonometry, died in Boston on October 22, 1914, at the age of 86 years. From 1886 to 1910 he was headmaster of the Boston Latin School.

MICHAEL A. MCGINNIS, who was the author of a book in which he claimed (falsely, of course) to have devised a universal solution for both numerical and literal equations, died recently in Kansas City. It is claimed by many who knew him intimately that, but for strong drink and dishonest business methods, he might have made a noteworthy record as a mathematician.

J. M. GREENWOOD, who, for many years and to the time of his death late last autumn, was superintendent of schools in Kansas City, Mo., was an ardent friend of the MONTHLY, having been a regular subscriber for a long period. He was a lover of mathematics for its own sake, spending many of his spare hours not only in mathematical recreations but in genuine mathematical study. For years he had been chairman of the appropriations committee of the National Education Association, where he always showed a keen appreciation of all investigations which had to do with improvement of mathematical teaching, one of his liberal recommendations being for the appropriation of funds to finance the work of the National Committee of Fifteen on Geometry Syllabus.

Professor GIOVAN BATTISTA GUCCIA, of the University of Palermo, died on October 29, 1914. He was the founder of the Circolo Matematico di Palermo and editor of its official publication the *Rendiconti*.

The Division of Mathematics of Harvard University announces that hereafter two appointments will be made each year to the Benjamin Peirce Instructorships in mathematics, which carry a stipend of $1,000 to $1,200 and allow the incumbents to pursue courses for higher degrees while giving instruction to the extent of about ten and one half hours per week. These instructors may be reappointed but not for more than three years.

"The Training of Mathematics Teachers" is the title of a paper read by Professor G. A. MILLER at the 1914 meeting of the Central Association of Science and Mathematics Teachers. The paper appears in the January, 1915, issue of *School Science and Mathematics*. A review of this stimulating address, together with a report of other features of this meeting will appear in a later issue of the MONTHLY.

Miss OLIVE C. HAZLETT, a graduate student at the University of Chicago, is the author of an article on "Invariantive characterization of some linear associative algebras" which appeared in the *Annals of Mathematics* for September, 1914.

Dr. THOMAS E. MASON, who took his doctorate at Indiana University last year, is instructor in mathematics at Purdue University. His thesis, entitled "Character of the solution of certain functional equations," was published in the *American Journal of Mathematics* for October, 1914.

Professor G. A. MILLER, of the University of Illinois, published an article in *The Popular Science Monthly* for November, 1914, which the readers of the AMERICAN MATHEMATICAL MONTHLY will be glad to see. It is entitled "Recent mathematical activities." It is an excellent résumé of matters with which everyone interested in mathematics should be familiar.

"The Uses for Mathematics" is the title of an article in *Science* of November 13, 1914, by Professor S. G. BARTON of Flower Observatory, University of Pennsyl-

vania, in which the great debt of the sciences to mathematics is dwelt upon. A list is given of 104 titles from the last edition of the *Encyclopedia Britannica* the treatment of which requires the infinitesimal calculus. The article closes with the following quotation from Sir John Herschel:

> "Admission to its sanctuary (the sanctuary of astronomy) and to the privileges and feelings of a votary is only to be gained by one means—sound and sufficient knowledge of mathematics, the great instrument of all exact inquiry, without which no man can ever make such advances in this or in any other of the higher departments of science as can entitle him to form an independent opinion on any subject of discussion within their range."

The Association of Mathematics Teachers of New Jersey is a new organization which held its first meeting November 7, 1914, with an attendance of about sixty. While composed chiefly of high-school teachers the organization is not to be conducted, we understand, so largely along so-called "normal" lines as has been the case in many instances, but rather the *common* interests of high school and college mathematics are to be conserved and the programs are to contain more papers dealing with subject matter of at least as high a grade as the calculus. To this end, besides several papers of the usual type, two of the newer type were given at this initial meeting, namely: on "Number and the Quadratic" by Professor RICHARD MORRIS of Rutgers College, and on "The Mechanics of Aviation" by Professor L. P. EISENHART of Princeton University. These papers were reported as admirably well adapted to the end desired and the resulting effect seemed most satisfactory. This is a step along a line similar to that undertaken by the California teachers as reported in this issue of the MONTHLY. We commend the California reading course to the New Jersey association. A moving spirit in this organization is HARRISON E. WEBB, of the Central High School, Newark, N. J.

The winter meetings of the American Mathematical Society held in Chicago on December 28, 29, 1914, and in New York on January 1, 2, 1915, were among the most enthusiastic and largely attended in many years. At Chicago there were sixty-one members present and twenty-two papers read. At New York there were ninety-five members present and thirty-two papers were read. At each meeting a dinner was held on the evening of the first day and thus opportunity was offered for social intercourse which was enjoyed to the full. These meetings have grown to be most inspiring occasions and are well worth the effort and expense of even long journeys in order to attend them. The attendance in New York was the largest ever recorded at a meeting of the Society, and the ten or more members from the West who helped to swell this number were amply repaid in the coin of friendly greeting, extended acquaintance, and mutual fellowship. The next Summer meeting of the Society will be in San Francisco early in August.

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THE HISTORY OF ZENO'S ARGUMENTS ON MOTION.

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

IV.

By FLORIAN CAJORI, Colorado College.

4. EARLY DISCUSSIONS OF LIMITS: GREGORY ST. VINCENT, GALILEO, HOBBES.

Limits in the Fifteenth and Sixteenth Centuries. With the fifteenth century new mathematical ideas appear. These germs are found in Greek philosophy, but they failed to develop during the dark centuries. In the fifteenth century the German cardinal, Nicolaus Cusanus (1401–1465), considered variability without being able to apply it successfully; he advanced the notion of a limit, though unable to pass correctly to the limit; he entertained the notion of infinitesimals but was not able to use them in an infinitesimal calculus.[1] He held that rules developed for the finite lose their validity for the infinite—a statement which later thinkers have not always heeded sufficiently. A point moving with infinite velocity in a circle is each moment in every position on the circle; hence it is at rest.

During the century, or century and a half, after Cusanus, concepts of limits and processes involving the passing to the limit begin to appear in different parts of Europe, like flowers on a field in early spring. Perhaps first in time, in the development of ideas considered by Cusanus, is Giovanno B. Benedetti, a distinguished forerunner of Galileo, who brought out a publication in 1585 at Turin, Italy. As early as 1586, and again in 1608, Simon Stevin at Leyden exhibited the process of passing to the limit.[2] In 1604 the Italian mathematician, Luc Valerio, published at Rome a treatise, *De centrogravitatis*, which contains a remarkable approach to the modern idea of limits.[3] In Galileo's celebrated

[1] K. Lasswitz, *Geschichte der Atomistik*, Hamburg und Leipzig, 1. Bd., 1890, pp. 283, 284, 287, See also Max Simon, "Cusanus als Mathematiker," *Festschr. H. Weber*, Leipzig und Berlin, 1912. pp. 298–337.

[2] H. Bosmans, "Sur quelques exemples de la méthode des limites chez Simon Stevin," *Annales de la société scientifique de Bruxelles*, T. 37, 1912–13, 2. fascicule.

[3] H. Bosmans, "Les démonstrations par l'analyse infinitésimale chez Luc Valerio," *Annales de la Société scientifique de Bruxelles*, T. 37, 1912–13, 2. fascicule; C. R. Wallner, "Ueber die Entstehung des Grenzbegriffes," *Bibliotheca mathematica*, 3. F., Bd. IV, 1903, p. 250.